But what is a Fourier Transform?

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For a lot of time now, I've struggled to understand the Fourier series, umm.. I mean Fourier analysis, wait no.. it's Fourier transform *innit?* Okay, then what does a DFT mean? What is FFT? What are epicycles and how can you draw any picture using a Fourier transform? ughh.. Do you see what I mean? There are so many definitions on the web regarding the Fourier and they aren't often very easy to differentiate, which makes learning a very daunting task for the uninitiated. The web is truly a scary place! This is my attempt to alleviate this confusion. Or maybe, nobody except Fourier or 3Blue1Brown can explain clearly what a Fourier is and I am just adding another meaningless blog post to this sea. Guess we'll find out!

Introduction

Fourier X is probably invented by Fourier, an 18th-century French mathematician and that's all I know about him. O wait, I also know that his full name is Jean-Baptiste Joseph Fourier. Ok that's all. End of funnies, now it's all serious ok?

To know Fourier, you must know what a Periodic function is!

As the name suggests, a periodic function is one that repeats itself. The time interval at which this repetition happens is called its *time-period*. Mathematically speaking, f is a periodic function if:

$$f(t+T) = f(T)$$

given that T is its time-period.

Trigonometric functions like *sine* and *cosine* are very good examples of periodic functions. Both of them have a period of 2π.

Is everything Sinusoids?

Thinking Velaciraptor face

thefouriertransformdotcom says that literally every function ever can be represented in the form of sinusoids *i.e., cosines* and *sines*. To appropriately approximate a function we might need an infinite number of sinusoids, but a finite and tractable number of sinusoids are enough to get a reasonable approximation. Therefore, what a Fourier X tries to do at a fundamental level is to decompose any function into a set of sinusoids. That's not too hard to remember is it? [1].

[1] Figure representing the superposition properties of a Fourier transform! For now look at the Figure 1

Let's start with Periodic functions

Let's say $f(x)$ is a periodic function. Think of a square wave! [2]
Our goal here is to sequester the given function into a set of sinusoids. We will first digress into something fun: Imagine how you'd represent any point in the 3-D space. We represent the space using 3 orthogonal lines *i.e.,* the X, Y and Z axes and represent every point as a vector. e.g., $(2,3,5)$. Notice that there are many possible ways to represent a point but most of these ways aren't Complete. For the system to be able to represent all the points, the system has to be made using orthogonal vectors.
Now, think about the *space* of functions. We need a set of orthogonal functions to form a system that can represent any function that can possibly be defined. We find out that the *sines* and *cosines* are actually orthogonal to each other when they aren't chilling in the trigonometric table. Consider the functions $\sin(mx)$ and $\cos(nx)$. It turns out that

$$\int_{-\pi}^{\pi} sin(mx)cos(nx)\ dx = 0$$

This is a nice result because, we can see that within the period of $-\pi \rightarrow \pi$, the above functions are orthogonal. Well, sorta, it's just loosely defined. Or I define it this way, so don't @ me.
Also, notice that the integral

$$\int_{-\pi}^{\pi} sin(mx)sin(nx)dx = \begin{cases} 0 & m \neq n \\ \pi & m = n \end{cases}$$

Therefore, it is safe to say that with some scaling the functions, $\sin(mx) and \cos(nx)$ are orthogonal. Similarly $\sin(mx) and \sin(nx)$ are orthogonal when $m \neq n$. Same results apply for cos as well. Therefore, we have a strong orthogonal system to represent functions.

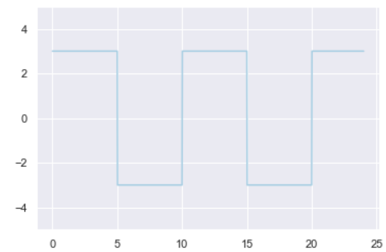

Figure 1: Square Wave

Representing a periodic function with a Series

Now that we have orthogonal components, we can represent any function as a weighted sum of these components. Mathematically,

$$f(x) = a_0 + \sum_{n=0}^{\infty} a_n \sin(nx) + \sum_{m=0}^{\infty} b_m \cos(mx) \tag{1}$$

Drawing analogues from vector calculus, the coefficient of each component *i.e.,* a_n, b_m can be calculated as

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\ \sin(nx)\ dx$$

This essentially is the Fourier series expansion of a periodic function. Note that eq(1) can only represent a periodic function with a period of 2π. For generic periods of say T, we write the equation follows:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \sin(\frac{2nx\pi}{T}) + \sum_{m=1}^{\infty} b_m \cos(\frac{2mx\pi}{T}) \tag{2}$$

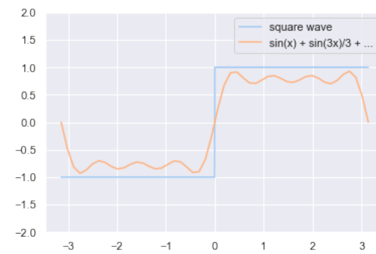

Figure 2: Square Wave

I would leave the task of finding the expressions of the coefficients as a task. This task is so pristine you'd never think of touching it. I know.

I am bored of using sinusoids and thinking about their orthogonality. And two separate σs are getting to my nerves. Can I unify them both? Yes! Can you think of any other orthogonal basis? Turns out the complex exponential $e^{\frac{i2\pi nt}{T}}$ form an orthogonal basis for the values of n ranging from $-\infty \to \infty$. In other words:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{i2\pi nt}{T}} \tag{3}$$

Again, the onus is once again on you to find the expression for the coefficients c_n.

Ok, but what if the function isn't Periodic?

In this case, you'd take an interval and obtain a Fourier series for that function in that interval only, by considering the function is periodic with that interval as the time period.

Fourier Transform

Till now, we have studied about obtaining the Fourier 'series' of a function by expressing it as a sum of orthogonal basis and their corresponding coefficients. Now, we will talk about a Fourier 'transform'. As the name suggests, it "transforms" a function from time domain to the frequency domain and *vice versa*. Now, let's say we were given a function in the time domain, $g(t)$. How would we find the corresponding coefficients of this function for a given frequency ω_0 in the frequency domain (Note that we are using ω_0 as frequency whereas in physics it is normally used to represent the). For example, consider the case where the given function $g(t)$ is a combination of two or more signals of different frequencies. How would we go about finding the constituent frequencies of that signal? We cull out the corresponding coefficients for a given ω_0. We know that $\omega_0 = \frac{2\pi}{T}$. Substituting this in the eq (3)

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t} \tag{4}$$

Now, if we need the value of the coefficient of a given ω, all we need to do is

$$\mathcal{F}(i\omega) = \int_{-\infty}^{\infty} f(t) e^{-in\omega_0 t} dt \tag{5}$$

But, what if a function is $zero$ outside a given interval $[-T, T]$? In that case, we will adjust the integration boundaries as follows:

$$\mathcal{F}(i\omega) = \frac{1}{2T} \int_{-T}^{T} f(t) e^{-i\omega t} dt \tag{6}$$

We will try to complete the equation with the Inverse Fourier transform, which is essentially going from time domain to frequency domain.

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}(\omega) e^{i\omega t} d\omega \tag{7}$$

Note that the scale factors differ if we are doing transforms in terms of frequency f instead of ω.

Then what the heck is a Discrete Fourier Transform?

Notice that the Fourier transform in the frequency domain is continuous i.e. it is defined for every value of ω. Let's try to simplify. What if it is discrete? Question, how do we make a continuous thing discrete? – Simple, we sample it! Let's sample N equally spaced points in the time domain and N equally spaced points in the frequency domain.

Let the points in the time domain be $f(0), f(1), f(n), ...f(N-1)$ and the points in the frequency domain are $\mathcal{F}(0), \mathcal{F}(1), \mathcal{F}(k), ..., \mathcal{F}(N-1)$. The conversion looks as follows:

$$\mathcal{F}(k) = \frac{1}{N} \sum_{n=0}^{N-1} f(n) e^{-i\frac{2\pi}{N}nk} \tag{8}$$

$$f(n) = \frac{1}{N} \sum_{k=0}^{N-1} \mathcal{F}(k) e^{-i\frac{2\pi}{N}nk} \tag{9}$$

References